1. The discrete random variables X and Y have joint PMF given by the table

$p_{X,Y}(x,y)$	$y=1$	$y=2$
$x=0$	$\frac{1}{10}$	$\frac{3}{10}$
$x=1$	$\frac{2}{10}$	$\frac{4}{10}$

(a) Find the marginal PMF of Y, presenting it as a table.
(b) Find the conditional PMF of X given $Y=2$, presenting it as a table.
(c) Are X and Y independent? Justify your answer.

Solution. (a) We complete the joint PMF table by summing along the rows and columns:

$p_{X,Y}(x,y)$	$y=1$	$y=2$	$p_X(x)$
$x=0$	$\frac{1}{10}$	$\frac{3}{10}$	$\frac{4}{10}$
$x=1$	$\frac{2}{10}$	$\frac{4}{10}$	$\frac{6}{10}$
$p_Y(y)$	$\frac{3}{10}$	$\frac{7}{10}$	1

(b) By the definition of conditional probability,

$$P(X=x \mid Y=2) = \frac{p_{X,Y}(x,2)}{p_Y(2)} = \begin{cases} \frac{3/10}{7/10} = \frac{3}{7}, & x=0, \\ \frac{4/10}{7/10} = \frac{4}{7}, & x=1, \\ 0, & \text{otherwise.} \end{cases}$$

(c) No. Since $P(X=0) = \frac{4}{10}$ and $P(Y=1) = \frac{3}{10}$,

$$P(X=0)P(Y=1) = \frac{4}{10} \cdot \frac{3}{10} = \frac{12}{100} \neq \frac{1}{10} = P(X=0, Y=1).$$

Hence X and Y are not independent.

□

2 (Conjugate priors)**.** A continuous random variable X is said to follow the Pareto distribution Pareto(α, c) if its density is

$$f_X(x) = \frac{\alpha c^\alpha}{x^{\alpha+1}} 1_{\{c \leq x\}}$$

where $\alpha, c > 0$. Suppose that $X \sim$ Pareto(α, c) and that $Y \mid X \sim$ Uniform$(0, X)$. Find the joint density (X, Y) and the conditional density $X \mid Y$.

Solution. We can obtain the joint density by using the product rule

$$\begin{aligned} f_{X,Y}(x,y) &= f_{Y|X}(y \mid x) f_X(x) \\ &= \frac{1}{x} 1_{\{0 \leq y \leq x\}} \cdot \frac{\alpha c^\alpha}{x^{\alpha+1}} 1_{\{c \leq x\}} \\ &= \frac{\alpha c^\alpha}{x^{\alpha+2}} 1_{\{\max(c,y) \leq x\}}. \end{aligned}$$

The conditional $X \mid Y = y$ can be found by using Bayes' theorem

$$f_{X|Y}(x \mid y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{1}{f_Y(y)} \cdot \frac{\alpha c^\alpha}{x^{\alpha+2}} 1_{\{\max(c,y) \le x\}} \tag{1}$$

where

$$f_Y(y) = \int_c^\infty f_{X,Y}(x,y)\, dx.$$

Luckily, we don't actually have to compute this integral because $f_Y(y)$ is constant in x. Moreover, we can easily tell that the the conditional density $f_{X|Y}(x \mid y)$ is proportional to the Pareto$(\alpha+1, \max\{c,y\})$ density, and every other term in (1) is simply absorbed into the normalizing constant. That is to say, $X \mid Y \sim \text{Pareto}(\alpha+1, \max\{c, Y\})$. □

3. Show that the knowledge of conditional distribution $X \mid Y$ and $Y \mid X$ is sufficient to characterize the joint (X, Y). That is to say, obtain the joint density $f_{X,Y}(x,y)$ given the conditional densities $f_{X|Y}(x|y)$ and $f_{Y|X}(y|x)$.

Solution. We approach this by trying to find $f_Y(y)$. Notice that once we have obtained this, we automatically obtain the joint by applying the product rule $f_{X,Y}(x,y) = f_Y(y) f_{X|Y}(x \mid y)$.

Recall Bayes' theorem

$$f_{X|Y}(x \mid y) = \frac{f_{Y|X}(y \mid x) f_X(x)}{f_Y(y)}$$

which implies that

$$\frac{f_{X|Y}(x \mid y)}{f_{Y|X}(y \mid x)} = \frac{f_X(x)}{f_Y(y)}$$

integrating over x results in

$$\int_{-\infty}^{\infty} \frac{f_{X|Y}(x \mid y)}{f_{Y|X}(y \mid x)}\, dx = \int_{-\infty}^{\infty} \frac{f_X(x)}{f_Y(y)}\, dx = \frac{1}{f_Y(y)}$$

meaning that

$$f_{X,Y}(x,y) = f_Y(y) f_{X|Y}(x \mid y) = f_{X|Y}(x \mid y) \cdot \left(\int_{-\infty}^{\infty} \frac{f_{X|Y}(x' \mid y)}{f_{Y|X}(y \mid x')}\, dx' \right)^{-1}$$

which is what we wanted to find. □