

1. The discrete random variables X and Y have joint p.m.f.

$$p_{X,Y}(x,y) = \begin{cases} cxy & (x,y) \in \{(1,1), (2,2), (1,3)\} \\ 0 & \text{otherwise} \end{cases}$$

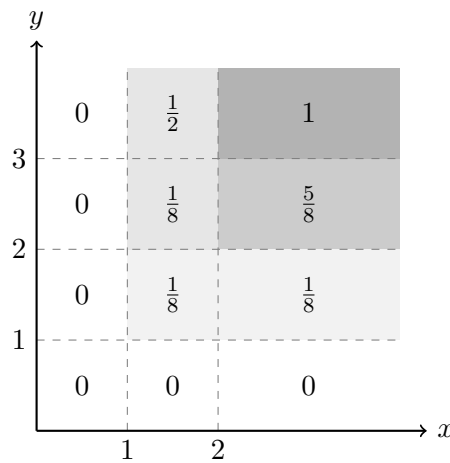
where c is a constant

- Find the value of c .
- Find the joint c.d.f.
- Find the marginal p.m.f.s of X and Y . Are they independent?

Solution. Clearly

$$1 = P(\mathbb{R}^2) = p_{X,Y}(1,1) + p_{X,Y}(2,2) + p_{X,Y}(1,3) = c(1 + 4 + 3) = 8c$$

which means that $c = 1/8$.



The joint c.d.f. is most easily found by looking at the support of the p.m.f. Recall that $F(x,y) = P(X \leq x, Y \leq y)$. Thus

$$F(x,y) = \begin{cases} 0 & x < 1, y < 1 \\ 1/8 & 1 \leq x < 2, 1 \leq y < 2 \\ 5/8 & x \geq 2, 2 \leq y < 3 \\ 1 & x \geq 2, y \geq 3 \end{cases}$$

The marginals are also easy to find. Note that the only values X can take are 1, 2. Thus

$$p_X(1) = p_{X,Y}(1,1) + p_{X,Y}(1,3) = \frac{4}{8}$$

$$p_X(2) = p_{X,Y}(2,2) = \frac{4}{8}$$

which is a valid p.m.f. Likewise, Y can only take on values in $\{1, 2, 3\}$. Thus

$$p_Y(1) = p_{X,Y}(1,1) = \frac{1}{8}$$

$$p_Y(2) = p_{X,Y}(2,2) = \frac{4}{8}$$

$$p_Y(3) = p_{X,Y}(1,3) = \frac{3}{8}$$

Note that X and Y are not independent, as

$$\frac{1}{8} = p_{X,Y}(1,1) \neq p_X(1)p_Y(1) = \frac{4}{8}$$

even though the p.m.f. $p_{X,Y}(x,y) = cxy$ would seem to indicate that X and Y are independent with marginals $p_X(x) = x, p_Y(y) = y$. $\square$

2. Let $D \subset \mathbb{R}^2$ be the region

$$D = \{(x,y) \in \mathbb{R}^2 : x \geq 1, 0 \leq y \leq 1/x^2\}.$$

Consider the random variables X and Y with joint density

$$f_{X,Y}(x,y) = \begin{cases} c\sqrt{x} & (x,y) \in D \\ 0 & (x,y) \notin D \end{cases}$$

where c is constant.

- (a) Find the value of c .
- (b) Find the marginal density of Y .

Solution. To find the constant C we solve the equation

$$1 = P(\mathbb{R}^2) = \int_D f_{X,Y}(x,y) dx dy.$$

Expanding the integral results in

$$\begin{aligned} \int_D f_{X,Y}(x,y) dx dy &= \int_1^\infty \int_0^{1/x^2} c\sqrt{x} dy dx \\ &= \int_1^\infty c\sqrt{x} \left(\int_0^{1/x^2} dy \right) dx \\ &= c \int_1^\infty x^{-3/2} dx \end{aligned}$$

this latter integral can be computed just fine

$$\int_1^\infty x^{-3/2} = \left[2x^{-1/2} \right]_1^\infty = 2$$

meaning that $c = 1/2$.

To find the marginal density of Y , we have to be a bit careful. First, note that

$$D = \{(x,y) \in \mathbb{R}^2 : 0 \leq y \leq 1, 1 \leq x \leq 1/\sqrt{y}\}$$

This can be obtained by noting that $0 \leq y \leq 1/x^2 \leq 1$ and

$$0 \leq y \leq 1/x^2 \implies x^2 \leq 1/y < \infty$$

so $x \leq 1/\sqrt{y}$. Combining all this together we get

$$f_Y(y) = \frac{1}{2} \int_1^{1/\sqrt{y}} \sqrt{x} dx = \frac{1}{2} \left[\frac{2}{3} x^{3/2} \right]_1^{1/\sqrt{y}} = \frac{1}{3} (y^{-3/4} - 1)$$

$\square$