

1. Suppose that  $f : \mathbb{R} \rightarrow \mathbb{R}$  defined by  $f(x) = cx 1_{[0,1]}(x)$  is the probability density function of some random variable  $X$ .

- (a) Find the value of  $c$ .
- (b) Find the c.d.f. of  $X$ .
- (c) Find the mean of  $X$ .

*Solution.* The value of  $c$  must be picked to ensure that the area under  $f$  is equal to 1. The region below  $f$  is a triangle with area  $1/2$ , which means that  $c = 2$ . Likewise, the c.d.f.  $F(x)$  is equal to the area under  $f$  between 0 and  $x$ . Whenever  $0 \leq x \leq 1$ , this is a triangle with height  $f(x) = cx$  and base  $x$ , resulting in  $F(x) = cx^2/2 = x^2$ .

The mean involves an honest to God integral

$$\int_0^1 x f(x) dx = 2 \int_0^1 x^2 dx = \frac{2}{3}$$

which is what we wanted to find.  $\square$

2 (Permutations). A permutation of  $n$  elements  $\{1, \dots, n\}$  can be formally viewed as a bijection  $\sigma : \{1, \dots, n\} \rightarrow \{1, \dots, n\}$  where  $\sigma(i)$  denotes where element  $i$  has been sent to after the shuffling. For instance, if we obtain  $(3, 2, 1, 4)$  after shuffling  $(1, 2, 3, 4)$ , this corresponds to the bijection

$$\sigma(1) = 3, \quad \sigma(2) = 2, \quad \sigma(3) = 1, \quad \sigma(4) = 4.$$

An element  $i$  is said to be a “fixed-point” of a permutation  $\sigma$  if  $\sigma(i) = i$ . For instance, in the example above 2 and 4 are fixed-points.

- (a) Let the  $S_n$  be the set of permutations of  $n$  elements. How many elements are in  $S_n$ ? What is the uniform distribution on  $S_n$ ?
- (b) What is the probability that  $i$  is the fixed-point of a permutation?
- (c) What is the expected number of fixed-points of a permutation?

*Solution.* The number of permutations of  $n$  elements is  $n!$ , so the uniform probability distribution is given by the p.m.f.  $p(\sigma) = 1/n!$ .

To obtain a permutation with a fixed-point at  $i$ , we can just fix  $i$  and permute the remaining  $n - 1$  elements. Therefore the number of such permutations is  $(n - 1)!$  and hence

$$P(\{\sigma : \sigma(i) = i\}) = \frac{(n - 1)!}{n!} = \frac{1}{n},$$

which, notably, does not depend on  $i$ .

Denote by  $N_i$  the indicator of the event that the  $i$ th element is a fixed point

$$N_i = 1_{\{\sigma : \sigma(i) = i\}}$$

therefore the total number of fixed-points is

$$N = N_1 + \dots + N_n$$

and it follows by the linearity of the expectation that

$$E N = \sum_{i=1}^n E N_i = \sum_{i=1}^n P(\{\sigma : \sigma(i) = i\}) = 1$$

which is what we wanted to find.  $\square$

**3** (Layer-cake formula). Show that for any continuous random variable such that  $P(X \geq 0) = 1$  the expectation can be written as

$$E[X] = \int_0^\infty P(X > t) dt$$

*Solution.* Recall that  $P(A) = E[1_A]$ . Using this idea we write

$$P(X > t) = E[1_{\{X > t\}}] = \int_0^\infty 1_{\{x > t\}} f_X(x) dx$$

Thus

$$\int_0^\infty P(X > t) dt = \int_0^\infty \int_0^\infty 1_{\{x > t\}} f_X(x) dx dt$$

Applying Fubini's theorem to interchange the order of integration results in

$$\int_0^\infty \int_0^\infty 1_{\{x > t\}} f_X(x) dt dx = \int_0^\infty f(x) \int_0^\infty 1_{\{x > t\}} dt dx = \int_0^\infty x f_X(x) dx$$

which is indeed the expectation  $E[X]$ .  $\square$

*Remark 1.* The above proof can be easily modified to work for *all* positive random variables, not just continuous ones, by sprinkling some measure theory magic. To be precise, we just need to replace all instances of  $f_X(x) dx$  with  $dP$ . Obviously you don't need to worry about the details, but if you try the next exercise you can assume that the layer-cake formula works for all random variables.

*Remark 2.* We can extend this to non-positive random variables by noting that  $X = X_+ - X_-$  where

$$X_- = \max(-X, 0) \geq 0, \quad X_+ = \max(X, 0) \geq 0$$

Thus

$$E X = E X_+ - E X_- = \int_0^\infty P(X_+ > t) dt - \int_0^\infty P(X_- > t) dt.$$

Indeed, this trick of writing  $X$  as the difference between two positive random variables is very useful in higher probability.

**4.** (Bonus continuation) Keeping the notation of the last problem,

(a) Show that for any  $n > 0$

$$E[X^n] = \int_0^\infty n t^{n-1} P(X > t) dt.$$

Hint: use  $u$ -substitution.

(b) Show that if for some  $c, \varepsilon > 0$  and  $t_0 > 0$ ,

$$P(|X| > t) \leq \frac{c}{t^{n+\varepsilon}}, \quad \text{for all } t \geq t_0$$

then  $E|X|^n < \infty$ .