Intermediate value theorem, numerical equation solving

Suppose we want to numerically solve the equation $f(x) = 0$ for $x \in [a, b]$. A reasonable assumption to make is that $f(a) < 0 < f(b)$. The intermediate value theorem asserts that a solution *exists,* but it doesn't tell us how to find it. We can look at the proof of the IVT for a lead, but all that says is that the solution is

$$c = \sup \{x \in [a, b] : f(x) < 0\}$$

which we still can't compute. The following two problems outline a new proof of the intermediate value theorem that's more useful for numerical purposes.

The following problem is needed in the proof, it's called the *nested interval theorem.*

1. (a) Suppose that A and B are two non-empty sets of numbers such that $x \le y$ for all $x \in A$ and $y \in B$. Show that $\sup A \le \inf B$.

(b) Consider a sequence of closed intervals $I_1 = [a_1, b_1], I_2 = [a_2, b_2], \ldots$. Suppose that $a_n \le a_{n+1}$ and $b_n \ge b_{n+1}$ for all n. Prove that there is a point x which is in every I_n.

Solution. Lets do part (a) first since we need it for part (b). Let $\alpha = \sup A$ and $\beta = \sup B$. Notice that every element of B is an upper bound of A. Therefore $\alpha \le b$ for all $b \in B$, and with similar reasoning we can conclude that $\beta \ge a$ for all $a \in A$. This means that α is a lower bound for B, and β is a lower bound for A. Thus $\alpha \le \beta$ since α is the *least* upper bound and β is the *greatest* lower bound.

For the second part, we need to show that $A = \{a_n : n \in \mathbb{N}\}$ and $B = \{b_n : n \in \mathbb{N}\}$ satisfy the hypotheses of part (a). That is, we have to demonstrate that $a_n \le b_m$ for any naturals n and m. This holds because

$$a_n \le a_{n+m} \le b_{n+m} \le b_m$$

and these inequalities hold because the intervals are nested within each other.

Therefore $\sup A \le \inf B$, meaning that the interval $[\sup A, \inf B]$ is non-empty, and is a subset of I_n for all n. This is what we wanted to show. □

2. (a) Suppose f is continuous $[a, b]$ and $f(a) < 0 < f(b)$. Consider the midpoint $m_1 = (a + b)/2$. Either,

- $f(m_1) = 0$,
- $f(m_1) > 0$, in which case $f(a)$ and $f(m_1)$ have different signs
- $f(m_1) < 0$, in which case $f(m_1)$ and $f(b)$ have different signs.

Use this to find an interval I_1 such that f has different signs on its endpoints. Repeat this process more times to find a nested sequence of intervals $I_1 \supset I_2 \supset \ldots$.

(b) Use the nested interval theorem to find a solution to the equation $f(x) = 0$.

Solution. We will specify the intervals I_n by their endpoints $[a_n, b_n]$, and let $m_n = (a_n + b_n)/2$ denote the midpoint of the nth interval. Let $I_0 = [a, b]$, and define I_{n+1} iteratively using I_n as follows

$$I_{n+1} = \begin{cases} \{m_n\} & f(m_n) = 0 \\ [a_n, m_n] & f(m_n)f(a_n) < 0 \\ [m_n, b_n] & f(m_n)f(b_n) < 0 \end{cases}$$

Notice that, in all cases, $I_{n+1} \subset I_n$. As such, the sets so constructed will satisfy $I_1 \supset I_2 \supset \dots$. The previous problem tells us that there is a point c that lies in all of these intervals.

Suppose for the sake of contradiction that $f(c) > 0$. This means that f is positive one some interval $J = (c - \delta, c + \delta)$ about c. Note that this interval has length 2δ. However, I_n has length $(b-a)/2^n$ so if we pick n big enough such that $(b-a)/2^n < \delta$, then $I_n = [a_n, b_n] \subset J$. Because of the way we constructed the I_n's, we know that f is negative at some point of I_n, contradicting that f is positive on J. Therefore $f(c) = 0$. $\square$

Some of you may have noticed that this is similar to the binary search algorithm for a list. The implementation on a computer is going to be similar.

3 (Bonus, numerical analysis)**.** Implement an algorithm based on the proof above that solves equations involving continuous functions. Use this algorithm to approximate the root of $x^3 - 3x + 1$. Your algorithm must allow the user to specify the tolerance value ε, and the output x of your algorithm must satisfy $|f(x)| < \varepsilon$.

If you are working with python, you can try and extend

```
def solve(f, a, b, tolerance):
    pass
```

The derivative

4. Find $f'(a)$ where (a) $f(x) = x^n$ for $n \geq 2$, (b) $f(x) = 1/x$ and (c) $f(x) = \sin x$.

Solution.

$$\begin{aligned} f'(a) &= \lim_{h\to 0} \frac{(a+h)^n - a^n}{h} \\ &= \lim_{h\to 0} \frac{\sum_{k=0}^{n} \binom{n}{k} a^{n-k}h^k - a^n}{h} \\ &= \lim_{h\to 0} \frac{na^{n-1}h + \binom{n}{2}a^{n-2}h^2 + \cdots}{h} \\ &= \lim_{h\to 0} \left(na^{n-1} + \binom{n}{2} a^{n-2}h + \cdots \right) \\ &= na^{n-1} \end{aligned}$$

$$
\begin{aligned}
f'(a) &= \lim_{h\to 0} \frac{\frac{1}{a+h} - \frac{1}{a}}{h} \\
&= \lim_{h\to 0} \frac{a-(a+h)}{a(a+h)h} \\
&= \lim_{h\to 0} \frac{-1}{a(a+h)} \\
&= -\frac{1}{a^2}
\end{aligned}
$$

$$
\begin{aligned}
f'(a) &= \lim_{h\to 0} \frac{\sin(a+h) - \sin a}{h} \\
&= \lim_{h\to 0} \frac{\sin a(\cos h - 1) + \cos a \sin h}{h} \\
&= \sin a \cdot \lim_{h\to 0} \frac{\cos h - 1}{h} + \cos a \cdot \lim_{h\to 0} \frac{\sin h}{h} \\
&= \cos a
\end{aligned}
$$

□

5. What are the slopes of the tangents to the curves $f(x) = 1/x$ and $g(x) = x^2$ at the point of of their intersection? Find the angle between these tangents.

Solution. Where do these curves intersect?

$$x^2 = \frac{1}{x} \implies x^3 = 1 \implies x = 1$$

Using the results from a previous question we conclude that $f'(1) = 2$ and $g'(1) = -1$.

The angle of inclination of the first line is $\theta = \arctan 2$, and the second line is $\phi = \arctan(-1)$. Therefore the angle between the two lines is $\arctan(2) - \arctan(-1) = \arctan(2) + \arctan(1)$. □

6. Find $f'(0), f'(1)$, and $f'(2)$ where $f(x) = x(x-1)^2(x-2)^3$.

Solution.

$$
\begin{aligned}
f'(0) &= \lim_{h\to 0} \frac{f(h) - f(0)}{h} \\
&= \lim_{h\to 0} \frac{h(h-1)^2(h-2)^3 - 0}{h} \\
&= \lim_{h\to 0} (h-1)^2(h-2)^3 \\
&= (0-1)^2(0-2)^3 \\
&= -8
\end{aligned}
$$

$$
\begin{aligned}
f'(1) &= \lim_{h\to 0} \frac{f(1+h)-f(1)}{h} \\
&= \lim_{h\to 0} \frac{(1+h)h^2(h-1)^3 - 0}{h} \\
&= \lim_{h\to 0} h(1+h)(h-1)^3 \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
f'(2) &= \lim_{h\to 0} \frac{f(2+h)-f(2)}{h} \\
&= \lim_{h\to 0} \frac{(2+h)(1+h)^2h^3 - 0}{h} \\
&= \lim_{h\to 0} (2+h)(1+h)^2h^2 \\
&= 0
\end{aligned}
$$

$\square$