Limits review

For the first few questions I'd like to review limits since we didn't get to spend much time on them last week.

1. *State the formal definition of the limit* $\lim_{x\to a^+} f(x) = L$ *and prove that* $\lim_{x\to 2^+} \sqrt{x-2} = 0$.

Solution. The formal definition of limit from the right is that for all $\varepsilon > 0$ there is a $\delta > 0$ such that $0 < |f(x) - L| < \varepsilon$ whenever $0 < x - a < \delta$.

Whenever approaching a problem like this, it's worth trying to bound *the error in the image*, i.e., $|f(x) - L|$, by an expression involving *the error in the input*, which is $|x - a|$. This is because once you have done this, you can use that expression to find a suitable δ. In this case, the error in the output is $|\sqrt{x-2} - 0|$, and we can observe that

$$|\sqrt{x-2}| = \frac{|x-2|}{\sqrt{x-2}}.$$

Since this is a limit from the right, then $x > a$, which means that $\sqrt{x-2} > \sqrt{a-2}$. If we replace the denominator with this, then we will be making it smaller, which will make the fraction larger. Thus

$$|\sqrt{x-2}| = \frac{|x-2|}{\sqrt{x-2}} < \frac{|x-2|}{\sqrt{a-2}}$$

This is a nice estimate of how far $\sqrt{x-2}$ is from zero, in terms of how far away x is from 2.

How do we use this to find δ? Well notice that if we make sure that x is within $\varepsilon \cdot \sqrt{a-2}$ distance units of 2, i.e., if $0 < x - a < \varepsilon \cdot \sqrt{a-2}$, then we will have

$$|\sqrt{x-2}| < \frac{|x-2|}{\sqrt{a-2}} < \varepsilon$$

which is what we wanted! Therefore, if $\delta = \varepsilon \cdot \sqrt{a-2}$, given any $\varepsilon > 0$, we will have $0 < |\sqrt{x-2} - 0| < \varepsilon$ whenever $0 < x - a < \delta$. $\square$

2. *Suppose that* $f(x) = g(x)$ *whenever* $0 < |x - a| < r$ *for some* r. *Show that* $\lim_{x\to a} f(x) = \lim_{x\to a} g(x)$ *if one of them exists.*

Solution. This means that the limit is a "local" property. It only cares about the behavior of the function in some "neighborhood" of a.

Suppose without loss of generality that $\lim_{x\to a} f(x) = L$ exists. This means that, given any $\varepsilon > 0$, there is a $\delta_1 > 0$ such that $0 < |f(x) - L| < \varepsilon$ whenever $0 < |x - a| < \delta_1$.

We'd like all the x's we're working with to be "close enough" to a so that $f(x) = g(x)$, *and* that $0 < |f(x) - L| < \varepsilon$. The first is achieved when x is within r distance units of a, and th second is achieved when x is within δ_1 units of a. This means that x is "close enough", when it lies within both of these intervals; that is, $0 < |x - a| < \min\{\delta_1, r\}$.

Since x lies within r units of a, then $f(x) = g(x)$, so

$$0 < |g(x) - L| = |f(x) - L|,$$

and since x is within δ_1 units of a, then

$$0 < |g(x) - L| = |f(x) - L| < \varepsilon$$

this means that for all $\varepsilon > 0$, there is $\delta := \min\{\delta_1, r\} > 0$ such that $0 < |g(x) - L| < \varepsilon$. This shows that $\lim_{x\to a} g(x) = L$ exists and is equal to the limit of f. $\square$

3. *Prove that for the function*

$$h(x) = \begin{cases} x, & x \in \mathbb{Q} \\ 0, & x \notin \mathbb{Q} \end{cases}$$

the limit of $h(x)$ *as* $x \to 0$ *will be* 0.

Solution. Let $\varepsilon > 0$ and let $\delta = \varepsilon$, then if $0 < |x| < \delta$ then $|h(x)| \leq |x| < \varepsilon$. $\square$

Supremum and infimum, basic consequences of completeness.

Certain problems on your assignment will be related to the concepts of supremum, infimum and the completeness of the real numbers. The next few problems are meant to develop some intuition on these concepts.

Recall that a set of real numbers A is said to be *bounded above* if there is a real number x such that $x \geq a$ for all $a \in A$, and *bounded below* if there is a number x such that $x \leq a$ for all $a \in A$. The *axiom of completeness*, also known as the *least upper bound property,* states that every set that's bounded above has a *least upper bound*, denoted by $\sup A$. To be precise, $y = \sup A$ if

- y is an upper bound of A
- If x is an upper bound of A then $y \leq x$.

Similarly, every set that's bounded below has a *greatest lower bound* $\inf A$.

An equivalent fact to the least upper bound property is the *Archimedean property*

4. *Show that the set of natural numbers* $\mathbb{N}$ *is not bounded above, and that for any* $\varepsilon > 0$ *there is a natural number* n *such that* $1/n < \varepsilon$.

Solution. Suppose that $\mathbb{N}$ is bounded above, then the least upper bound property says that it has a least upper bound $s = \sup \mathbb{N}$. This means that if n is any natural number, then $s \geq n + 1 \geq n$ since $n + 1 \in \mathbb{N}$. This implies that $s - 1 \geq n$ for all $n \in \mathbb{N}$, meaning that $s - 1$ is *also* an upper bound of $\mathbb{N}$, but this means that s is no longer the *smallest* upper bound, since $s - 1$ is smaller. This is a contradiction, therefore $\mathbb{N}$ is not bounded above.

The second part follows from this pretty easily. Since $\mathbb{N}$ is not bounded above, there is a natural number $n > 1/\varepsilon$ so $1/n < \varepsilon$. $\square$

A consequence of this fact is the so called *density of the rationals.*

5. *(a) Suppose that $y - x > 1$. Prove that there is an integer k such that $x < k < y$. (b) Suppose $x < y$, prove that there is a rational number r such that $x < r < y$.*

Solution. Let $A = \{n \in \mathbb{N} : n < x\}$ and let $m = \inf A$, this means that $m \leq x$ but $m + 1 > x$. This means that $m + 1 \leq x + 1 < y$ so $x < k = m + 1 < y$ which is what we wanted.

For the second part, let n be a number such that $1/n < y - x$. This means that $ny - nx > 1$. By the first part this means that there is a rational number m such that $nx < m < ny$. Therefore $x < m/n < y$ and $r = m/n$ is what we wanted to find. $\square$

6. *Show that if f is continuous and $f(x + y) = f(x) + f(y)$ for all x and y then $f(x) = cx$ for some c and for all $x \in \mathbb{R}$.*

Solution. We proved in a previous week that if $f(x + y) = f(x) + f(y)$ then $f(x) = cx$ for all rational x. This time we will use continuity and the density of the rationals to show that $f(x) = cx$ for *all* real numbers.

Notice that if $|f(a) - ca| < \varepsilon$ for all $\varepsilon > 0$, then $f(a) = ca$ (why?) So we will show this. Let $\varepsilon > 0$ be arbitrary and notice that

$$|f(a) - ca| = |f(a) - f(x) + f(x) - ca| \leq |f(a) - f(x)| + |f(x) - ca|$$

The $|f(a) - f(x)|$ should tell us that we should use continuity about here. So let δ be given such that $|f(a) - f(x)| < \varepsilon/2$ if $|x - a| < \delta$. The previous problem tells us that there is a rational number x such that $|x - a| < \min\{\varepsilon/2|c|, \delta_1\}$, which means that

$$|f(a) - ca| \leq |f(a) - cx| + |cx - ca| < \varepsilon/2 + |c|\varepsilon/2|c| = \varepsilon$$

showing that $f(a) = ca$ for all $a \in \mathbb{R}$. $\square$

Continuity

The following problems are about continuity.

7. *Prove that if f is continuous, then so is $g(x) = |f(x)|$.*

Solution. Since f is continuous, then for all $\varepsilon > 0$ there is a $\delta_1 > 0$ such that $|f(x) - f(a)| < \varepsilon$ whenever $|x - a| < \delta_1$. It follows from the second triangle inequality that

$$||f(x)| - |f(a)|| \leq |f(x) - f(a)| < \varepsilon$$

so we let $\delta = \delta_1$ and the desired result follows. $\square$

8. *Suppose that f is continuous at $g(a)$ and g is continuous at a. Show that $f \circ g$ is continuous at a.*

Solution. Given an arbitrary $\varepsilon > 0$, let δ_1 be given such that if $|y - g(a)| < \delta_1$ then $|f(y) - f(g(a))| < \varepsilon$, then let δ_2 be given such that if $|x - a| < \delta_2$ then $|g(x) - g(a)| < \delta_1$.

Suppose that $|x - a| < \delta_2$, then $|g(x) - g(a)| < \delta_1$, which means that $|f(g(x)) - f(g(a))| < \varepsilon$, which is what we wanted to show. $\square$

9. *Show that if f is continuous at a and $f(a) > 0$, then there is an open interval centered at a on which f is positive.*

10 (Bonus, topology of $\mathbb{R}$)**.** *Let $B_r(a) = \{x \in \mathbb{R} : |x - a| < r\} = (a-r, a+r)$ denote the open "ball" of radius r about a. A subset $U \subset \mathbb{R}$ is called **open** if every point $x \in U$ has an open ball $B_r(x)$ which is a proper subset of U. Show that f is continuous if and only if $f^{-1}(U)$ is open for every open U.*