Computing derivatives

1 (Logarithmic differentiation)**.** Differentiate

$$f(x) = \sqrt[x]{x}, \quad g(x) = \sqrt{\frac{x(x-1)}{x-2}}, \quad h(x) = (\cos x)^{\sin x}$$

2. Show that the function $f(x) = xe^{-x^2/2}$ satisfies the differential equation $xf'(x) = (1-x^2)f(x)$

3 (Bonus)**.** Calculate the 100th derivative of the function

$$\frac{x^2+1}{x^3-x}$$

Solution. Note that

$$\frac{x^2+1}{x^3-x} = \frac{x^2+1}{x(x-1)(x+1)}$$

From last week, we know that reciprocals of linear functions are simple to evaluate. Lets try to write this as a sum of such functions

$$\frac{x^2+1}{x(x-1)(x+1)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1} = \frac{(A+B+C)x^2+(B-C)x-A}{x(x-1)(x+1)}$$

so A, B, and C must satisfy

$$\begin{cases} A+B+C=1 \\ B-C=0 \\ -A=1 \end{cases}$$

Solving this system results in $A=-1, B=C=1$. Thus

$$\frac{x^2+1}{x^3-x} = -\frac{1}{x} + \frac{1}{x-1} + \frac{1}{x+1}$$

The nth derivative of these three are

$$\frac{d^n}{dx^n}\left(-\frac{1}{x}\right) = \frac{(-1)^{n+1}n!}{x^{n+1}}, \quad \frac{d^n}{dx^n}\left(\frac{1}{x-1}\right) = \frac{(-1)^n n!}{(x-1)^{n+1}}, \quad \frac{d^n}{dx^n}\left(\frac{1}{x+1}\right) = \frac{(-1)^n n!}{(x+1)^{n+1}}$$

Therefore the 100th derivative of the given function is

$$-\frac{100!}{x^{101}} + \frac{100!}{(x-1)^{101}} + \frac{100!}{(x+1)^{101}}$$

□

4. Suppose $f(x) = x^5 + 2x^3 + 7x - 4$. Find $(f^{-1})'(6)$.

Solution. We know that $(f^{-1})'(6) = f'(f^{-1}(6))^{-1}$. Note that

$$f(1) = 1+2+7-4 = 6$$

so $f^{-1}(6) = 4$. Moreover,

$$f'(x) = 5x^4 + 6x^2 + 7$$

so $f'(f^{-1}(6))^{-1} = 1/f'(4) = 1/(5\cdot 256 + 6\cdot 16 + 7)$. □

5. Suppose that f is continuous and differentiable on $[0,1]$. You know, as you demonstrated in the homework and the midterm, that there is a point $x \in [0,1]$ such that $f(x) = x$. Show that there is only one such point of $f'(t) \neq 1$ for all $0 \leq t \leq 1$.

Optimization

6 (Geometrical optics)**.** In the 17th century, the lawyer and mathematician Pierre de Fermat observed that when light goes from point A to point B it always takes the path of least time. Suppose that A and B lie in two different media separated by a plane. The speed of light in these media is v_1 and v_2 respectively. If θ_1 is the angle of incidence, and θ_2 is the angle of refraction, show that $v_2 \sin\theta_1 = v_1 \sin\theta_2$

Solution. See Optical Demonstrations □

7 (Regression)**.** Suppose we have collected some data $\{(x_1, y_1), \ldots, (x_n, y_n)\}$ from some experiment, and we want to find the line of best fit that passes through the origin. That is, we want to find a function $f_\beta(x) = \beta x$, that best fits our data. We measure the "fit" of this function by the mean squared error:

$$E(\beta) = \sum_{i=1}^{n} (y_i - f_\beta(x_i))^2.$$

find the value of β that minimizes E.

Solution. Lets say that the amount of error we incur from the ith data point is $L_i = (y_i - \beta x_i)^2$, so $E = \sum_i L_i$. Since $L_i' - 2(y_i - \beta x_i)x_i$, we have that

$$E' = \sum_i L_i' = -2\sum_i (y_i - \beta x_i)x_i = -2\left(\sum_i x_i y_i - \beta \sum_i x_i^2\right)$$

Therefore the optimal value $\hat{\beta}$ must satisfy

$$E'(\hat{\beta}) = -2\left(\sum_i x_i y_i - \hat{\beta}\sum_i x_i^2\right) = 0$$

will be given by

$$\hat{\beta} = \frac{\sum_i x_i y_i}{\sum_i x_i^2}$$

and this is indeed a global minimum since $E''(\hat{\beta}) = \sum_i x_i^2 > 0$. Bonus: give a geometric interpretation for the value $\hat{\beta}$. □

8 (Bonus, regression continued)**.** Suppose we collect the data $\{x_1, \ldots, x_n\}$ from repeated trials of the same experiment with measurement errors. We want to report a single number x that best represents/fits this data. Find x if the lack of fit is measured by (a) $E(x) = \sum_{i=1}^{n}(x_i - x)^2$ and (b) $E(x) = \sum_{i=1}^{n} |x_i - x|$. Calculus isn't going to help much with the second one.

9. Find the side lengths of the largest rectangle that can be inscribed in the ellipse $E : x^2/a^2 + y^2/b^2 = 1$.

Solution. Let $(x, y) \in E$ be the top-right vertex of the rectangle with width $2x$ and height $2y$. The area, therefore, is $A = 4xy$. We wish to maximize A subject to the constraint that (x, y) lies on the ellipse: $x^2/a^2 + y^2/b^2 = 1$. Fortunately, since (x, y) is in the top-right quadrant of the plane, we can solve for y as a function of x.

$$y = b\sqrt{1 - x^2/a^2} = \frac{b}{a}\sqrt{a^2 - x^2}$$

which allows us to the area as a single-variable function of x,

$$A(x) = 4xy = \frac{4b}{a} x\sqrt{a^2 - x^2}$$

We have successfully reduced the problem to something we already know how to do.

To optimize A, we have to find the value x that satisfies $A'(x) = 0$ and $A''(x) < 0$. Thus, (differentiation omitted)

$$A'(x) = \frac{4b}{a} \frac{a^2 - 2x^2}{\sqrt{a^2 - x^2}} = 0, \quad A''(x) = \frac{4bx(2x^2 - 3a^2)}{a(a^2 - x^2)^{3/2}} < 0$$

From the first we can deduce that $x = a/\sqrt{2}$. Notice that since $a > x$, we can say that $2x^2 - 3a^2 < 0$, $(a^2 - x^2)^{3/2} > 0$, and $4bx/a > 0$, thus the second derivative $A''(x) <$ as wanted. This shows that $x = a/\sqrt{2}$, $y = b/\sqrt{2}$ produce the rectangle of largest area. The sidelengths, therefore, must be $\sqrt{2}a, \sqrt{2}b$.

$\square$

Graphing

10. Sketch the graph of $f(x) = 4x^{\frac{1}{3}} + x^{\frac{4}{3}}$. Carefully indicate (1) domain, (2) intercepts, (3) symmetry, (4) asymptotes, (5) derivatives, (6) critical points, (7) points of inflection.

Solution. (1) The domain is obviously $\mathbb{R}$.

(2) Intercepts. The only y-intercept is $(0,0)$ For the x-intercepts:

$$0 = 4x^{1/3} + x^{4/3} = x^{1/3}(4 + x) \implies x = 0, -4.$$

Thus $(-4,0)$ and $(0,0)$.

(3) Symmetry. Note that

$$f(-x) = 4(-x)^{1/3} + (-x)^{4/3} = -4x^{1/3} + x^{4/3} \neq f(x)$$

and also $f(-x) \neq -f(x)$. So f is neither even nor odd.

(4) Asymptotes. There are obviously not vertical asymptotes Slant asymptotes:

$$\frac{f(x)}{x} = 4x^{-2/3} + x^{1/3} \to \infty \quad (x \to \infty),$$

so none. Similarly, there are no horizontal asymptotes as $f(x) \to \pm\infty$ as $x \to \pm\infty$

(5) Derivatives.

$$f'(x) = 4 \cdot \frac{1}{3} x^{-2/3} + \frac{4}{3} x^{1/3} = \frac{4 + 4x}{3x^{2/3}} = \frac{4(x+1)}{3x^{2/3}}.$$

$$f''(x) = \frac{4(x-2)}{9x^{5/3}}.$$

(6) Critical points. Solve $f'(x) = 0$:

$$4(x+1) = 0 \Rightarrow x = -1.$$

Also $f'(x)$ is undefined at $x = 0$, and $0 \in \mathrm{Dom}(f)$. Thus the critical points are $x = -1$ and $x = 0$.

7. Inflection Points. Solve $f''(x) = 0$:

$$x - 2 = 0 \Rightarrow x = 2.$$

$f''(x)$ is undefined at $x = 0$, which must also be checked. Thus the candidates are $x = 0$ and $x = 2$.

8. Sign Chart.

	$(-\infty,-1)$	-1	$(-1,0)$	0	$(0,2)$	2	$(2,\infty)$
$f(x)$	$\nearrow$	min	$\nearrow$	IP	$\nearrow$	IP	$\nearrow$
$f'(x)$	$+$	0	$+$	DNE	$+$		$+$
$f''(x)$	$+$		$+$	DNE	$-$	0	$+$

$\square$