More on the derivative

1. Find $f'(0)$ if

$$f(x) = \begin{cases} g(x)\sin(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

and $g(0) = g'(0) = 0$.

Computing derivatives

2. If f is differentiable at a, let $d(x) = f(x) - f'(a)(x-a) - f(a)$. Find $d'(a)$.

3 (Logarithmic differentiation)**.** Differentiate

$$f(x) = \sqrt[x]{x}, \quad g(x) = \sqrt{\frac{x(x-1)}{x-2}}, \quad h(x) = (\cos x)^{\sin x}$$

4. Show that the function $f(x) = xe^{-x^2/2}$ satisfies the differential equation $xf'(x) = (1-x^2)f(x)$

5. The radius of a sphere is increasing at a uniform rate of 5 cm/sec. At what rate are the area of the surface of the sphere and the volume of the sphere increasing when the radius becomes 50 cm.

6. At what point of the curve $y^2 = 2x^3$ is the tangent perpendicular to the straight line $4x - 3y + 2 = 0$?

7. Show that the hyperbolas $xy = a^2$ and $x^2 - y^2 = b^2$ intersect at a right angle.

8. Find the nth derivative of $1/x$.

9 (Bonus)**.** Calculate the 100th derivative of the function

$$\frac{x^2+1}{x^3-x}$$

10. A point M is in motion around the circle $x^2 + y^2 = a^2$ with constant angular velocity ω. Find the velocity and acceleration of the of the projection M_1 on the x-axis.

Significance of the derivative

11. In the 17th century, the lawyer and mathematician Pierre de Fermat observed that when light goes from point A to point B it always takes the path of least time. Suppose that A and B lie in two different media separated by a plane. The speed of light in these media is v_1 and v_2 respectively. If θ_1 is the angle of incidence, and θ_2 is the angle of refraction, show that $v_2 \sin\theta_1 = v_1 \sin\theta_2$

12. Show that for $0 \le x < y < 2\pi$ the inequality

$$|\sin x - \sin y| \le |x - y|$$

holds.

13. Suppose we have collected some data $\{(x_1, y_1), \ldots, (x_n, y_n)\}$ from some experiment, and we want to find the line of best fit that passes through the origin. That is, we want to find a function $f_\lambda(x) = \lambda x$ that best fits our data. We measure the "fit" of this function by the mean squared error:

$$E(\lambda) = \frac{1}{n} \sum_{i=1}^{n} (y_i - f_\lambda(x_i))^2.$$

find the value of λ that minimizes E.

14. Find the side lengths of the largest rectangle that can be inscribed in the ellipse $x^2/a^2 + y^2/b^2 = 1$.