We wish to show that if f is integrable on $[a, b]$, $[a, c]$, and $[c, b]$, then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

using only the theory of Riemann sums. No Darboux integeration allowed.

Proof. Let $\{x_i\}_{i=1}^n$ be a Riemann partition of the interval $[a, c]$ with $\Delta x = x_i - x_{i-1}$. We want to define a Riemann partition $\{y_i\}_1^m$ on $[c, b]$ where the length of each subinterval is $\Delta y = \Delta x$. However, since the magnitudes $b-c$ and Δx are not guaranteed to be commensurable, we also have to account for a small error term ϵ. It follows from the Archimedean property that m is the unique number such that $(b - c) = m\Delta x + \epsilon$ and $0 \leq \epsilon < \Delta x$.

For convenience, let $\{t_i\}_1^{n+m}$ be a Riemann partition of the interval $[a, b - \varepsilon]$ where $t_i = x_i$ for $0 \leq i \leq n$ and $t_{n+i} = y_i$ for $0 \leq i \leq m$. Notice that $t_n = x_n = y_0$.

Since m and ϵ where defined (indirectly) in terms of n using the Archimdean property, and $m\Delta x + \epsilon = (b - c)$, then

$$\int_b^c f(x)\,dx = \lim_{n\to\infty} \left(\sum_{i=1}^m f(y_i^*)\Delta x + f(y_{m+1}^*)\epsilon \right),$$

where $y_{m+1}^* \in [y_m, y_m + \epsilon]$. Therefore

$$\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \lim_{n\to\infty} \left(\sum_{i=1}^n f(x_i^*)\Delta x + \sum_{i=1}^m f(y_i^*)\Delta x + f(y_{m+1}^*)\epsilon \right).$$

Since we defined $t_i = x_i$ for $0 \leq i \leq n$ and $t_{n+i} = y_i$ for $0 \leq i \leq m$ and $\Delta t = \Delta x = \Delta y$ then this expression can be rewritten as

$$\lim_{n\to\infty} \left(\sum_{i=1}^n f(t_i^*)\Delta t + \sum_{i=n+1}^{n+m} f(t_i^*)\Delta t + f(y_{m+1}^*)\epsilon \right),$$

which, by Σ-properties, is equal to

$$\lim_{n\to\infty} \left(\sum_{i=1}^{n+m} f(t_i^*)\Delta t + f(y_{m+1}^*)\epsilon \right).$$

Since, $\Delta x \to 0$ as $n \to \infty$, and $\epsilon < \Delta x$, then $\epsilon \to 0$ as well. So the sum of the two integrals is equal to

$$\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n+m} (f(t_i^*)\Delta t) = \int_a^b f(x)\,dx.$$

Which is what we wanted to show. Q.E.D.