Problem 1. *Using multiple sequences in the style of the Fibonacci sequence, we have generated the triangle of numbers below. Each number is the sum of the two numbers above it and the first two numbers in the nth column are the nth and $n-1$th Fibonacci numbers respectively.*

$$\begin{array}{cccccccc}
0 & 1 & 1 & 2 & 3 & 5 & 8 & \cdots \\
 & 0 & 1 & 1 & 2 & 3 & 5 & \cdots \\
 & & 2 & 3 & 5 & 8 & 13 & \cdots \\
 & & & 4 & 7 & 11 & 18 & \cdots \\
 & & & & 12 & 19 & 31 & \cdots \\
 & & & & & 30 & 49 & \cdots \\
 & & & & & & 80 & \cdots \\
 & & & & & & & \ddots
\end{array}$$

We form a new sequence, $\{a_k\}_{k\geq 0}$ from the numbers in the hypotenuse of this triangle.

$$0, 0, 2, 4, 12, 30, 80, \cdots$$

Show that:

$$a_k = 2(a_{k-1} + a_{k-2}) - a_{k-3}$$

Solution
First thing that we can do is each column to turn this triangle into a rectangle with coordiantes. We'll also ignore the outer row and column for convenience. The diagonal is bolded.

	0	1	2	3	4	5	$\cdots$
0	**0**	1	1	2	3	5	$\cdots$
1	1	**2**	3	5	8	13	$\cdots$
2	1	3	**4**	7	11	18	$\cdots$
3	2	5	7	**12**	19	31	$\cdots$
4	3	8	11	19	**30**	49	$\cdots$
5	5	13	18	31	49	**80**	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Lets also introduce some notation for dealing with this rectangle. We let $\binom{a}{b}$ be the number at the ath column and the bth row. Now we can make some general observations.
Now we can restate our problem with our new notation. $\{a_k\}$ would then be $\{\binom{k}{k}\}$, and then the recurrence relation would be

$$\binom{k}{k} = 2\left(\binom{k-1}{k-1} + \binom{k-2}{k-2}\right) - \binom{k-3}{k-3}$$

First we can notice that the rectangle is symmetric in relation to the diagonal. We can state this as a lemma.

Lemma 1. *The rectangle given above symmetric in relation to the diagonal and that all rows as well as columns are recurrence relations. Otherwise stated as*

$$\binom{a}{b} = \binom{a-1}{b} + \binom{a-2}{b} = \binom{a}{b-1} + \binom{a}{b-2} = \binom{b}{a}$$

Proof. We can observe from the rectangle that these properties hold for at least some values of a and b. So we'll assume *both* as our inductive hypothesis. Then notice that because all columns are recurrence relations,

$$\begin{aligned}\binom{a+1}{b} &= \binom{a}{b} + \binom{a-1}{b} \\ &= \binom{b}{a} + \binom{b}{a-1} \\ &= \binom{b}{a+1}\end{aligned}$$

Which means that

$$\begin{aligned}\binom{a+2}{b} &= \binom{a+1}{b} + \binom{a}{b} \\ &= \binom{b}{a+1} + \binom{b}{a}\end{aligned}$$

Which completes our strong induction. $\square$

So we want to find a formula for $\binom{k}{k}$. We can use lemma 1 to see that

$$\begin{aligned}\binom{k}{k} &= \binom{k}{k-1} + \binom{k}{k-2} \\ &= \binom{k-1}{k-1} + \binom{k-2}{k-2} + \binom{k-2}{k-1} + \binom{k-1}{k-2}\end{aligned}$$

We can use lemma 1 again to get values for:

$$\begin{aligned}\binom{k-2}{k-1} &= \binom{k-1}{k-1} + \binom{k-3}{k-1} \\ \binom{k-1}{k-2} &= \binom{k-2}{k-2} + \binom{k-3}{k-2}\end{aligned}$$

Plugging these in we get

$$\binom{k}{k} = 2\left(\binom{k-1}{k-1} + \binom{k-2}{k-2}\right) + \binom{k-3}{k-2} - \binom{k-3}{k-1}$$

Now notice that

$$\binom{k-3}{k-1} = \binom{k-3}{k-2} + \binom{k-3}{k-3}$$

So we can solve this identity for $\binom{k-3}{k-3}$ to get

$$\binom{k-3}{k-3} = \binom{k-3}{k-1} - \binom{k-3}{k-2}$$

Plugging this in our equation we get that

$$\binom{k}{k} = 2\left(\binom{k-1}{k-1} + \binom{k-2}{k-2}\right) - \binom{k-3}{k-3}$$

Which is what we wanted. $\square$