Continuity

1. *Show that if f is continuous and $f(x+y) = f(x) + f(y)$ for all x and y then $f(x) = cx$ for some c and for all $x \in \mathbb{R}$.*

Solution. We proved in a previous week that if $f(x+y) = f(x) + f(y)$ then $f(x) = cx$ for all rational x. This time we will use continuity and the density of the rationals to show that $f(x) = cx$ for *all* real numbers.

Notice that if $|f(a) - ca| < \varepsilon$ for all $\varepsilon > 0$, then $f(a) = ca$ (why?) So we will show this. Let $\varepsilon > 0$ be arbitrary and notice that

$$|f(a) - ca| = |f(a) - f(x) + f(x) - ca| \leq |f(a) - f(x)| + |f(x) - ca|$$

The $|f(a) - f(x)|$ should tell us that we should use continuity about here. So let δ be given such that $|f(a) - f(x)| < \varepsilon/2$ if $|x - a| < \delta$. The previous problem tells us that there is a rational number x such that $|x - a| < \min\{\varepsilon/2|c|, \delta_1\}$, which means that

$$|f(a) - ca| \leq |f(a) - cx| + |cx - ca| < \varepsilon/2 + |c|\varepsilon/2|c| = \varepsilon$$

showing that $f(a) = ca$ for all $a \in \mathbb{R}$. □

2. *Suppose that f is continuous at $g(a)$ and g is continuous at a. Show that $f \circ g$ is continuous at a.*

Solution. Given an arbitrary $\varepsilon > 0$, let δ_1 be given such that if $|y - g(a)| < \delta_1$ then $|f(y) - f(g(a))| < \varepsilon$, then let δ_2 be given such that if $|x - a| < \delta_2$ then $|g(x) - g(a)| < \delta_1$.

Suppose that $|x - a| < \delta_2$, then $|g(x) - g(a)| < \delta_1$, which means that $|f(g(x)) - f(g(a))| < \varepsilon$, which is what we wanted to show. □

3 (Bonus, topology of $\mathbb{R}$)**.** *Let $B_r(a) = \{x \in \mathbb{R} : |x - a| < r\} = (a-r, a+r)$ denote the open "ball" of radius r about a. A subset $U \subset \mathbb{R}$ is called **open** if every point $x \in U$ has an open ball $B_r(x)$ which is a proper subset of U. Show that f is continuous if and only if $f^{-1}(U)$ is open for every open U.*

Solution. Lets begin with the forward direction. Suppose that f is continuous, and that U is an open set. This means that there is an open ball $B_\varepsilon(f(a)) \subset U$. The continuity of f means that there is a $\delta > 0$ such that if $0 < |x - a| < \delta$ then $|f(x) - f(a)| < \varepsilon$, that is, if $x \in B_\delta(a)$ then $f(x) \in B_\varepsilon(f(a)) \subset U$. This means that $B_\delta(a) \subset f^{-1}(U)$, and hence $f^{-1}(U)$ is open.

Now for the backwards direction, suppose that whenever U is open then $f^{-1}(U)$ is open. We wish to show that for all $\varepsilon > 0$ there is $\delta > 0$ such that $x \in (a - \delta, a + \delta)$ implies $f(x) \in (f(a) - \varepsilon, f(a) + \varepsilon)$. Let $U = (f(a) - \varepsilon, f(a) + \varepsilon)$ which is an open set. This means that $f^{-1}(U)$ is an open set. Since $a \in f^{-1}(U)$, then there is an open ball $B_\delta(a) = (a-\delta, a+\delta)$ which is a proper subset of $a \in f^{-1}(U)$. All this means, is that if $x \in (a - \delta, a + \delta)$, then $f(x) \in U = (f(a) - \varepsilon, f(a) + \varepsilon)$ which is what we wanted to show. □

The intermediate value theorem

4. *Suppose that f is a continuous function on $[0,1]$ and that $f(x)$ is in $[0,1]$ for each x. Prove that $f(x) = x$ for some number x.*

Solution. Since $f(0) \in [0,1]$, then either $f(0) = 0$ or $f(0) > 0$. In the first case we would be done so we consider the second case. Similarly, either $f(1) = 1$ or $f(1) \in [0,1)$. Again, there is nothing more to do in the first case. What remains is to check the case where $f(0) \in (0,1]$ and $f(1) \in [0,1)$. Consider $g(x) = f(x) - x$. Note that $g(0) > 0$ and $g(1) < 0$ which means that, by the intermediate value theorem, there is a point $y \in (0,1)$ such that $g(y) = 0$. That is, $f(y) = y$. $\square$

5. *Find on all functions which are continuous on $[a,b]$ and which only take on rational values.*

Solution. Intuitively you should be able to guess that these are constant functions. Suppose that f is not constant, i.e., there are numbers x and y in $[a,b]$ such that $f(x) < f(y)$. This means that f must take on every number between $[f(x), f(y)]$ for z between x and y. As we showed last week, there must be a irrational number in $[f(x), f(y)]$ which means that there is a point z between x and y such that $f(z) \notin \mathbb{Q}$. This shows that f does not only take rational values. $\square$

Evaluation of limits

6. *Evaluate*

$$\lim_{x\to 2} \frac{x^2-4}{x^2-3x+2}, \quad \lim_{x\to 1}\left(\frac{1}{1-x} - \frac{3}{1-x^3}\right)$$

Solution. For the first one, we factor the numerator and denominator,

$$\lim_{x\to 2} \frac{x^2-4}{x^2-3x+2} = \lim_{x\to 2} \frac{(x+2)(x-2)}{(x-2)(x-1)} = \lim_{x\to 2} \frac{x+2}{x-1} = 4$$

Second one has a similar strategy

$$\begin{aligned}
\lim_{x\to 1}\left(\frac{1}{1-x} - \frac{3}{1-x^3}\right) &= \lim_{x\to 1}\left(\frac{1}{1-x} - \frac{3}{(1-x)(1+x+x^2)}\right) \\
&= \lim_{x\to 1} \frac{x^2+x-2}{(1-x)(1+x+x^2)} \\
&= \lim_{x\to 1} \frac{(x-1)(x+2)}{(1-x)(1+x+x^2)} \\
&= \lim_{x\to 1} \frac{-(x+2)}{1+x+x^2} \\
&= -1
\end{aligned}$$

$\square$

The following problem is meant to illustrate the technique of using a continuous substitution in a limit.

7. *Evaluate*

$$\lim_{x\to 0}\frac{\sqrt{1+x}-1}{\sqrt[3]{1+x}-1}$$

Solution. Let

$$f(x)=\frac{x^3-1}{x^2-1}$$

and notice that if $g(x)=\sqrt[6]{1+x}$ then

$$f(g(x))=\frac{(1+x)^{\frac{3}{6}}-1}{(1+x)^{\frac{2}{6}}-1}=\frac{\sqrt{1+x}-1}{\sqrt[3]{1+x}-1}.$$

Since g is continuous at 0 and it equal to $g(0)=1$, we have that

$$\lim_{x\to 0}f(g(x))=\lim_{x\to g(0)}f(x)=\lim_{x\to 1}\frac{x^3-1}{x^2-1}=\lim_{x\to 1}\frac{x^2+x+1}{x+1}=\frac{3}{2}$$

□

Bonus: compactness and the extreme value theorem

Let U be a subset of $\mathbb{R}$. A collection $\mathcal{O}=\{A_i : i\in I\}$ is set to be an *open cover* of U if each of the A_i's is open and

$$U\subset\bigcup_{i\in I}A_i.$$

For instance, the sets $\mathcal{O}=\{A_i=(i,i+2): i\in\mathbb{N}\}$ form an infinite open cover for $U=(2,3)$. since they are all open and

$$U\subset A_1\cup A_2\cup\cdots=\bigcup_{i\in\mathbb{N}}A_i=(1,\infty).$$

We can also have finite open covers, an example is

$$\mathcal{O}'=\{A_i=(i,i+2): i=1,2,3\}.$$

In fact, since every set in $\mathcal{O}'$ is also in $\mathcal{O}$, then $\mathcal{O}'$ is said to be a *finite subcover* of $\mathcal{O}$. A set U is said to be *compact* if every open cover of U has a finite subcover.

As an example, lets show that $U=(2,3)$ is not compact. Notice that

$$U=\bigcup_{n\geq 0}(2,3-10^{-n})=(2,2)\cup(2,2.9)\cup(2,2.99)\cup\cdots=(2,3)$$

However, if we remove even a single one of these sets then their union will not cover U.

8. *Show that for any $a,b\in\mathbb{R}$, the closed interval $[a,b]$ is compact. Hint: your proof should be similar to the proof of the intermediate value theorem.*

A set U is said to be *closed* if $\mathbb{R}-U$ is open. It is said to be *bounded* if it is a subset of some closed interval $[a,b]$.

9 (The Heine-Borel Theorem)**.** *Show that if U is closed and bounded then it is compact. Hint: use the compactness of $[a,b]$.*