1. *Show that if x is irrational then $\sqrt{x}$ is irrational.*

Solution. The contrapositive of this statement is that if $\sqrt{x}$ is rational then x is rational. That is pretty easy to show: suppose that $\sqrt{x} = p/q$, then $x = p^2/q^2$, which is also rational! $\square$

2. *for which numbers $a, b, c,$ and d does the function $f(x) = (ax+b)/(cx+d)$ satisfy $f(f(x)) = x$ for all x in the domain of f?*

Solution. if

$$x = f(f(x)) = \frac{a\left(\dfrac{ax+b}{cx+d}\right)+b}{c\left(\dfrac{ax+b}{cx+d}\right)+d} = \frac{(a^2+bc)x+(ab+bd)}{(ac+cd)x+(bc+d^2)}$$

for all x, then,

$$(ac+cd)x^2 + (d^2-a^2)x - (ab+db) = 0.$$

since this holds for all x, then we must have

$$(a+d)c = d^2 - a^2 = (a+d)b = 0.$$

There are three cases at this point, either $a = d = 0$, $a = d$ or $a = -d$.

Case 1. If $a = d = 0$, then b and c can be anything and we have $f(x) = b/(cx)$.

Case 2. If $a = d$ and both are non-zero, then $b = c = 0$ so $f(x) = ax/d = x$.

Case 3. If $a = -d$, then $f(x) = (ax+b)/(cx-a)$ with domain $x \neq a/c$. note that $f(x)$ must always be in the domain of f, so it can never be equal to a/c. As such

$$\frac{ax+b}{cx-a} \neq \frac{a}{c},$$

which implies that $a^2 + bc \neq 0$. $\square$

3. *Write the limit $\lim_{x\to-\infty} f(x) = -\infty$ as a formal statement involving N and M, and sketch a graph that illustrates the roles of these constants.*

Solution. The formal statement of this limit is that for all $M > 0$, there is an $N > 0$ such that $f(x) < -M$ whenever $x < -N$. $\square$

4. *Show that $\lim_{x\to1} 3x = 3$ by following the steps outlined in exercise 68 of 1.2.*

Solution. Given $\varepsilon > 0$, let $\delta = \varepsilon/3$, and suppose that $0 < |x-1| < \delta$. This means that

$$0 < |x-1| < \frac{\varepsilon}{3} \implies |3x-3| < \varepsilon$$

which is what we wanted to show. $\square$

5. *Show that $\lim_{x\to0^+} 1/x = \infty$ by following the steps outlined in exercise 70 of 1.2.*

Solution. Given $M > 0$, let $\delta = 1/M$, and suppose that $0 < x < \delta$. This means that

$$\frac{1}{x} > \frac{1}{\delta} = M$$

which is what we wanted. $\square$

6. *State the formal definition of the limit* $\lim_{x\to a^+} f(x) = L$ *and prove that* $\lim_{x\to 2^+} \sqrt{x-2} = 0$.

Solution. The formal definition of limit from the right is that for all $\varepsilon > 0$ there is a $\delta > 0$ such that $0 < |f(x) - L| < \varepsilon$ whenever $0 < x - a < \delta$.

Whenever approaching a problem like this, it's worth trying to bound *the error in the image*, i.e., $|f(x) - L|$, by an expression involving *the error in the input*, which is $|x - a|$. This is because once you have done this, you can use that expression to find a suitable δ. In this case, the error in the output is $|\sqrt{x-2} - 0|$, and we can observe that

$$|\sqrt{x-2}| = \frac{|x-2|}{\sqrt{x-2}}.$$

Since this is a limit from the right, then $x > a$, which means that $\sqrt{x-2} > \sqrt{a-2}$. If we replace the denominator with this, then we will be making it smaller, which will make the fraction larger. Thus

$$|\sqrt{x-2}| = \frac{|x-2|}{\sqrt{x-2}} < \frac{|x-2|}{\sqrt{a-2}}$$

This is a nice estimate of how far $\sqrt{x-2}$ is from zero, in terms of how far away x is from 2.

How do we use this to find δ? Well notice that if we make sure that x is within $\varepsilon \cdot \sqrt{a-2}$ distance units of 2, i.e., if $0 < x - a < \varepsilon \cdot \sqrt{a-2}$, then we will have

$$|\sqrt{x-2}| < \frac{|x-2|}{\sqrt{a-2}} < \varepsilon$$

which is what we wanted! Therefore, if $\delta = \varepsilon \cdot \sqrt{a-2}$, given any $\varepsilon > 0$, we will have $0 < |\sqrt{x-2} - 0| < \varepsilon$ whenever $0 < x - a < \delta$. $\square$

7. *Show that the function* $f(x) = x/|x|$ *does not have a limit at* $x = 0$.

Solution. What does it mean for f to have a limit at 0? It means that there is some real number L such that for all $\varepsilon > 0$, there is a $\delta > 0$ where $0 < |f(x) - L| < \varepsilon$ whenever $0 < |x - a| < \delta$. The negation of this statement is that *for all real numbers L, there is some $\varepsilon > 0$ such that for all $\delta > 0$, there is an x with $0 < |x - a| < \delta$, but $|f(x) - L| \geq \varepsilon$.* In intuitive terms, if L is any number, then it cannot be well approximated by $f(x)$ no matter how much you push x towards a.

In this case, f can be written as

$$f(x) = \begin{cases} 1 & x > 0 \\ -1 & x < 0 \end{cases}$$

It should make intuitive sense that f doesn't have a limit when we look at the graph. If L is any number on the y-axis, then all $f(x)$ for negative x

will have an error of $|-1-L|$, whereas all the $f(x)$ for positive x will have an error of $|1-L|$; and we can't reduce this error by moving x closer to 0.

To make this precise, let L be given, and set $\varepsilon = \min\{|-1-L|, |1-L|\}$. If $\delta > 0$ is any positive number, then $x = \delta/2$ will satisfy $0 < |x-0| = \delta/2 < \delta$. However, notice that $|f(x) - L| = |1-L| \geq \min\{|-1-L|, |1-L|\} = \varepsilon$. This shows that f has no limit at $x = 0$. $\square$

8. *Show that if a function f cannot have two different limits approach a. That is, if f approaches L_1 near a, and approaches L_2 near a, then $L_1 = L_2$.*

Solution. Our strategy here will be to show that given any $\varepsilon > 0$, the distance between L_1 and L_2 will be less than ε. If the distance between two numbers is less than all positive numbers, then they must be equal!

Since this limit exists, then

1. Let δ_1 be given such that if $0 < |x-a| < \delta_1$ then $0 < |f(x) - L_1| < \varepsilon/2$.
2. Let δ_2 be given such that if $0 < |x-a| < \delta_1$ then $0 < |f(x) - L_2| < \varepsilon/2$.

Now lets try to estimate the distance between L_1 and L_2, that is, $|L_1 - L_2|$. Notice that this is the same as $|L_1 - f(x) + f(x) - L_2|$. It follows from the triangle inequality that

$$|L_1 - L_2| \leq |f(x) - L_1| + |f(x) - L_2,|$$

and this holds for all x. So if we only look at those x's which satisfy $0 < |x-a| < \min\{\delta_1, \delta_2\}$, then $0 < |f(x)-L_1| < \varepsilon/2$ and $0 < |f(x)-L_2| < \varepsilon/2$. That is, in this interval

$$|L_1 - L_2| \leq |f(x) - L_1| + |f(x) - L_2| < \varepsilon/2 + \varepsilon/2 < \varepsilon,$$

which is what we wanted to show. $\square$

9. *Suppose that $f(x) = g(x)$ whenever $0 < |x-a| < r$ for some r. Show that $\lim_{x\to a} f(x) = \lim_{x\to a} g(x)$ if one of them exists.*

Solution. This means that the limit is a "local" property. It only cares about the behavior of the function in some "neighborhood" of a.

Suppose without loss of generality that $\lim_{x\to a} f(x) = L$ exists. This means that, given any $\varepsilon > 0$, there is a $\delta_1 > 0$ such that $0 < |f(x) - L| < \varepsilon$ whenever $0 < |x-a| < \delta_1$.

We'd like all the x's we're working with to be "close enough" to a so that $f(x) = g(x)$, *and* that $0 < |f(x) - L| < \varepsilon$. The first is achieved when x is within r distance units of a, and th second is achieved when x is within δ_1 units of a. This means that x is "close enough", when it lies within both of these intervals; that is, $0 < |x-a| < \min\{\delta_1, r\}$.

Since x lies within r units of a, then $f(x) = g(x)$, so

$$0 < |g(x) - L| = |f(x) - L|,$$

and since x is within δ_1 units of a, then

$$0 < |g(x) - L| = |f(x) - L| < \varepsilon$$

this means that for all $\varepsilon > 0$, there is $\delta := \min\{\delta_1, r\} > 0$ such that $0 < |g(x) - L| < \varepsilon$. This shows that $\lim_{x \to a} g(x) = L$ exists and is equal to the limit of f. $\square$