In the following problems we will make use of the *fundamental theorem of arithmetic.* This states that any natural number n can be written uniquely as a product of primes. To be more precise, this theorem states that for any n there are primes $p_1, \ldots, p_m$ and other naturals $a_1, \ldots, a_m$ such that $n = p_1^{a_1} \ldots p_m^{a_m}$, and this can only be done in one way. Notice that if n is a perfect square, then the all of exponents in its prime factorization must be even.

Problem

Prove that $\sqrt{3}$ is irrational.

Suppose that $\sqrt{3} = p/q$, where p and q are coprime integers. This means that $3q^2 = p^2$, and hence p is a multiple of 3, $p = 3k$.

Why can we say that p is a multiple of 3? It should be clear that p^2 is a multiple of 3, and since it's a perfect square, then the exponent of 3 in its prime factorization must be even, so it must be at least 2. The exponent of 3 in p is half the exponent of 3 in the prime factorization of p^2, which means that it is at least one!

Therefore, $3q^2 = 9k^2$, implying that $q^2 = 3k^2$. This means that q is a multiple of 3, contradicting the assumption that p and q are coprime.

Bonus problem from last week

Prove that if x satisfies $x^n + a_{n-1}x^{n-1} + \cdots + a_0 = 0$ for some integers $a_{n-1}, \ldots, a_0$, then x is irrational unless x is an integer.

Suppose that $x = p/q$ where p and q are coprime integers. This means that

$$\frac{p^n}{q^n} + a_{n-1}\frac{p^{n-1}}{q^{n-1}} + \cdots + a_0 = 0$$

Multiplying both sides by q^n results in $p^n + a_{n-1}p^{n-1}q + \cdots + a_0 q^n = 0$. This obviously means that

$$p^n = -q(a_{n-1}p^{n-1} + \cdots + a_0 q^{n-1})$$

Unless $q = \pm 1$, then it must have some prime factor k, in which case p^n is divisible by k, and so is. However, this would contradict our assumption that and q share no divisors. This showing that unless $q = \pm 1$—i.e., unless x is an integer—then it must be irrational.

Again, why can we say that if a prime k divides p^n, then it divides p? Similar to what we said about perfect squares, all of the exponents in the prime factorization of p^n will be multiples of n. Since the exponent of k in p is $1/n$th the exponent of k in p^n, then it will still be greater than or equal to 1, which means that p is divisible by k, as wanted.

Homework problem 2, arithmetic and geometric means

Given two positive numbers $0 \leq a \leq b$ show that

$$a \leq \sqrt{ab} \leq \frac{a+b}{2} \leq b$$

Won't post solution here since its on the homework. We can extend this problem in two ways.

First extension of AM-GM, harmonic mean

The *harmonic mean* of a and b is $2ab/(a+b)$. Show that

$$\frac{2ab}{a+b} \leq \sqrt{ab} \leq \frac{a+b}{2}$$

Since we've already showing the AM-GM inequality, we can use that here. Lets multiply both sides by $\sqrt{ab}$. This results in

$$ab \leq \frac{(a+b)\sqrt{ab}}{2}$$

Dividing both sides by $(a+b)/2$ results in

$$\frac{2ab}{a+b} \leq \sqrt{a+b}$$

which is what we wanted to show.

Second extension of AM-GM, weighted mean

The weighted mean of n numbers $x_1 \leq \cdots \leq x_n$ with weights $w_1, \ldots, w_n$ is defined by

$$\frac{w_1 x_1 + \cdots + w_n x_n}{w_1 + \cdots + w_n}$$

Show that for any choice of weights, the weighted mean is larger than x_1 and smaller than x_n.

Notice that we can increase the fraction by replacing all the x_i's in the numerator with x_n.

$$\frac{w_1 x_1 + \cdots + w_n x_n}{w_1 + \cdots + w_n} \leq \frac{(w_1 + \cdots + w_n) x_n}{(w_1 + \cdots + w_n)} \leq x_n$$

and similarly

$$\frac{w_1 x_1 + \cdots + w_n x_n}{w_1 + \cdots + w_n} \geq \frac{(w_1 + \cdots + w_n) x_1}{(w_1 + \cdots + w_n)} \geq x_1$$

which is what we wanted to show.

An important fact in analysis is the fact that there is a rational number between any two real numbers. To prove this we need the following two facts:

1. For any $\varepsilon > 0$ there is a natural n such that $1/n < \varepsilon$.
2. If $x < y$ are any two real numbers with $|x - y| > 1$, then there's a natural number n between them.
   We can't really prove these facts yet (we actually can prove two but won't in the interest of time).

Density of the rationals

Between any two real numbers a and b there is a rational number r.

Let $\varepsilon = b - a$ (assuming without loss of generality that $b > a$). By the first fact above there is a natural number n such that $1/n < \varepsilon$. Let $x = na$ and $y = nb$, which means that $|x - y| = n|a - b| > 1$. By fact two, there's an a natural number m between x and y. Therefore $na = x < m < y = nb$, which means that $a < m/n < b$. Since $r = m/n$ is obviously rational, we have shown what we wanted.

Problem

Let

$$f(x) = \begin{cases} e^{\sin(\lfloor x \rfloor \pi)}, & x \in \mathbb{Q} \\ x^2, & x \notin \mathbb{Q} \end{cases}$$

1. Prove or disprove that $f(f(x)) = f(x)$.
2. Find $f^{-1}[\{0\}]$.
3. Find $f^{-1}[(0, 1/\sqrt{2})]$.
4. Find $f^{-1}[f^{-1}[\{1\}]]$.

To solve this problem, we introduce the notion of the *inverse image*. If $f : X \to Y$ is a function, and $A \subset Y$, then $f^{-1}[A]$ is defined to be the set of points $x \in X$ that are sent to Y by f. That is,

$$f^{-1}[A] = \{x \in X : f(x) \in A\}$$

This notion will be very useful in your upcoming mathematics and statistics courses.

1. Notice that $f(\sqrt{2}) = 2$, which is not equal to $f(2) = 1$. Therefore the given statement is false.
2. By the definition of the inverse image

$$
\begin{aligned}
f^{-1}[\{0\}] &= \{x \in \mathbb{R} : f(x) = 0\} \\
&= \{x \in \mathbb{Q} : e^{\sin(\lfloor x \rfloor \pi)} = 0\} \cup \{x \notin \mathbb{Q} : x^2 = 0\} \\
&= \emptyset
\end{aligned}
$$

3. Since $f = 1$ on $\mathbb{Q}$, then we only consider the problem outside of it. Thus

$$f^{-1}[(0, 1/\sqrt{2})] = \{x \notin \mathbb{Q} : 0 < x^2 < 2^{-1/2}\} = (0, 2^{-1/4}) - \mathbb{Q}$$

4. Since the square of no irrational number is 1, then we only consider this problem on the rationals. Note that $f^{-1}[\{1\}] = \mathbb{Q}$, so we need to find $f^{-1}[\mathbb{Q}]$. This is evidently $\{x \notin \mathbb{Q} : x^2 \in \mathbb{Q}\}$, that is, those irrational numbers whose squares are rational.