1. *Show that the function* $f(x) = \ln\left(x + \sqrt{1+x^2}\right)$ *is an odd function. Then, find the inverse function.*

Solution. Recall that f is called odd if $f(-x) = -f(x)$. Is this true about the given function?

$$f(-x) = \ln\left(-x + \sqrt{1+(-x)^2}\right) = \ln\left(-x + \sqrt{1+x^2}\right)$$

Is this really equal to $-f(x) = -\ln\left(x + \sqrt{1+x^2}\right)$? Notice that

$$\begin{aligned} e^{f(-x)} &= e^{\ln\left(-x+\sqrt{1+x^2}\right)} = -x + \sqrt{1+x^2} \\ e^{-f(x)} &= e^{-\ln\left(x+\sqrt{1+x^2}\right)} = \frac{1}{x+\sqrt{1+x^2}} \end{aligned}$$

We can show that these two are equal by multiplying and dividing by the conjugate:

$$\frac{1}{x+\sqrt{1+x^2}} \cdot \frac{-x+\sqrt{1+x^2}}{-x+\sqrt{1+x^2}} = \frac{-x+\sqrt{1+x^2}}{-x^2+(1+x^2)} = -x+\sqrt{1+x^2}.$$

This shows that $e^{f(-x)} = e^{-f(x)}$, and since the function $x \mapsto e^x$ is one-to-one, then $f(-x) = -f(x)$, which is what we wanted to show.

Lets find f^{-1}. Let $y = f^{-1}(x)$, then

$$f(f^{-1}(x)) = f(y) = \ln\left(y + \sqrt{1+y^2}\right) = x$$

We may solve this equation for y.

$$\begin{aligned} \ln\left(y + \sqrt{1+y^2}\right) &= x \\ e^{\ln\left(y+\sqrt{1+y^2}\right)} &= e^x \\ y + \sqrt{1+y^2} &= e^x \\ \sqrt{1+y^2} &= e^x - y \\ 1+y^2 &= (e^x-y)^2 \\ 1+y^2 &= e^{2x} - 2ye^x + y^2 \\ 2ye^x &= e^{2x} - 1 \\ y &= \frac{e^{2x}-1}{2e^x} \end{aligned}$$

which is what we wanted to find. □

2. *Sketch the graph of the function* $f(x) = \lfloor x + \lfloor x \rfloor \rfloor$. *Prove that, for all integers* n, $f(x) = 2n$ *if* $x \in [n, n+1)$.

Solution. Let n be an integer, and let $x \in [n, n+1)$. That is, $x = n + \varepsilon$, where $0 \le \varepsilon < 1$. Therefore $\lfloor x \rfloor = n$, and so $f(x) = \lfloor x+n \rfloor = \lfloor 2n+\varepsilon \rfloor = 2n$, which is what we wanted to show. □

3. *Prove that if x satisfies $x^n + a_{n-1}x^{n-1} + \cdots + a_0 = 0$ for some integers $a_{n-1}, \ldots, a_0$, then x is irrational unless x is an integer. (Hint: if q divides p^n, then q divides p).*

Solution. Suppose for the sake of contradiction that $x = p/q$, for some integers p and q with no common factors. Then

$$\frac{p^n}{q^n} + a_{n-1}\frac{p^{n-1}}{q^{n-1}} + \cdots + a_0 = 0$$

Multiplying both sides by q^n results in

$$p^n + a_{n-1}p^{n-1}q + \ldots a_0 q^n = 0$$

Meaning that $p^n = -q(a_{n-1}p^{n-1} + \cdots + a_0 q^{n-1})$, and hence p^n is divisible by q. This implies that p is divisible by q, which contradicts our initial assumption. (Exercise: prove that if q divides p^n, then q divides p). $\square$

4. *Prove that any function f with domain $\mathbb{R}$ can be written uniquely as $f = E + O$ where E is an even function and O is an odd function.*

Solution. Obviously we need to find E and O such that $f(x) = E(x)+O(x)$ for all $x \in \mathbb{R}$. Since E and O are supposed to be even and odd respectively, we must also have $f(-x) = E(-x) + O(-x) = E(x) - O(x)$. This results in the following system of equations

$$\begin{cases} f(x) = E(x) + O(x) \\ f(-x) = E(x) - O(x) \end{cases}$$

Solving this for $E(x)$ and $O(x)$ results in

$$E(x) = \frac{f(x) + f(-x)}{2}, \quad O(x) = \frac{f(x) - f(-x)}{2}$$

which is what we wanted to find. $\square$

5. *Suppose f satisfies $f(x + y) = f(x) + f(y)$ for all x and y. (a) Prove that $f(x_1 + \cdots + x_n) = f(x_1) + \cdots + f(x_n)$ for all $x_1, \ldots, x_n$. (b) Prove that for all rational x, $f(x) = cx$ for some c.*

Solution. For (a), note that

$$\begin{aligned} f(x_1 + (x_2 + \cdots + x_n)) &= f(x_1) + f(x_2 + (x_3 + \cdots + x_n)) \\ &= f(x_1) + f(x_2) + f(x_3 + \cdots + x_n) \\ &= \ldots \\ &= f(x_1) + f(x_2) + \cdots + f(x_n) \end{aligned}$$

Lets do (b). Let $x = p/q$ where p and q are integers. Since x is equal to $1/q$ added to itself p times, then $f(x)$ is equal to $f(1/q)$ added to itself p times, that is, $f(x) = pf(1/q)$. So what is $f(1/q)$? Notice that $qf(1/q) = f(1)$, so $f(1/q) = f(1)/q$. Therefore $f(x) = f(1)x$, so $c = f(1)$, as we wanted. $\square$

6. *For which numbers $a, b, c,$ and d does the function $f(x) = (ax+b)/(cx+d)$ satisfy $f(f(x)) = x$ for all x in the domain of f?*

Solution. If

$$x = f(f(x)) = \frac{a\left(\dfrac{ax+b}{cx+d}\right)+b}{c\left(\dfrac{ax+b}{cx+d}\right)+d} = \frac{(a^2+bc)x+(ab+bd)}{(ac+cd)x+(bc+d^2)}$$

for all x, then,

$$(ac+cd)x^2+(d^2-a^2)x-(ab+db)=0.$$

Since this holds for all x, then we must have

$$(a+d)c = d^2-a^2 = (a+d)b = 0$$

. There are three cases at this point, either $a=d=0$, $a=d$ or $a=-d$.

Case 1. If $a=d=0$, then b and c can be anything and we have $f(x)=b/(cx)$.

Case 2. If $a=d$ and both are non-zero, then $b=c=0$ so $f(x)=ax/d=x$.

Case 3. If $a=-d$, then $f(x)=(ax+b)/(cx-a)$ with domain $x \neq a/c$. Note that $f(x)$ must always be in the domain of f, so it can never be equal to a/c. As such

$$\frac{ax+b}{cx-a} \neq \frac{a}{c},$$

which implies that $a^2+bc \neq 0$. □